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An Optimization Model for Scheduling Withdrawals from Tax-Deferred Retirement Accounts

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As a growing number of Americans reach retirement age, more and more people are facing important decisions about how to withdraw savings from tax-deferred retirement accounts (TDRAs). These decisions are complicated by the Federal Tax Code which imposes a number of rules and regulations on these withdrawals. Since these decisions collectively involve billions of dollars, *the potential loss from even slightly suboptimal decision making is very large. In this paper, we present a mathematical programming model that can be used to assist retirees and/or their advisors in determining the optimal schedule of withdrawals from TDRAs.*

I. INTRODUCTION

As the general population of the United States ages, more and more Americans are having to make important decisions about how to withdraw money from tax-deferred retirement accounts (TDRAs). TDRAs include individual retirement accounts (IRAs), qualified corporate retirement plans, and tax-deferred salary reduction plans covered by Section 403(b) of the Internal Revenue Service (IRS) code. Generally speaking, TDRAs are an attractive investment alternative for two primary reasons: 1) they offer a way for some taxpayers to reduce their current tax liability; and 2) interest earnings on these investments are sheltered from taxes until they are withdrawn. Typically the argument is made that people should invest in TDRAs so their investment will grow faster (i.e., money is being compounded rather than taxed) and likely will be subjected to a lower tax rate at the taxpayer's retirement.

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American Demographics (1989) reports that investors have recently put as much as $100 billion annually into IRAs alone. Thus, the amounts and timing of the withdrawals from TDRAs can have a significant impact on the amount of taxes one pays and the accumulation of personal wealth. However, this decision is complicated by the Federal Tax Code which imposes a myriad of rules and regulations on these withdrawals. In addition to the normal income tax that applies to withdrawals from TDRAs, there are also penalty taxes associated with withdrawing *too much* or *too little* and making withdrawals *too soon* or *too late*. Coupled with these rules are a number of more subtle investment issues which further complicate the process of determining the optimum amount to withdraw each year.

In this paper we first briefly review the rules regarding distributions from TDRAs in order to demonstrate and motivate the need for an optimization model that can assist investors (or their advisors) in determining how to withdraw money from these accounts. Next, a mathematical programming model for this problem is proposed and described in detail. Finally, an example is provided to demonstrate the potential merit of the proposed model versus a number of heuristics that have been suggested in the literature.

II. THE PROBLEM

A complete description of the regulations regarding TDRA distributions is available in IRS publication 590 (1992) or less technical summary articles (e.g., Abramson, 1989; Katz, 1990; McCommbe, 1989). For the reader's convenience and reference, we have also summarized these rules in Table 1. It goes without saying that there are some minor exceptions to these rules.

Premature Distributions Tax

Congress has established various rules concerning when withdrawals can be made from TDRAs. Generally speaking, the tax law is written with the intention that withdrawals from TDRAs begin no earlier than age 59.5. With few exceptions, any money withdrawn from TDRAs before age 59.5 is subject to a 10 percent Premature Distribution Tax penalty in addition to regular income taxes.

TABLE 1.
Summary of TDRA Tax Rules
(*na* = not applicable)

	Age When Withdrawal is Made		
Rule	*Before 59.5*	*From 59.5 to 70.5*	*After 70.5*
Premature Distribution Tax[1]	10%	na	na
Minimum Distribution Tax[2]	na	na	50%
Excess Distribution Tax[3]	5%	15%	15%

Notes: [1]Applies to the total amount withdrawn in any year.
[2]Applies to the amount by which the minimum legally required withdrawal in any year exceeds the actual amount withdrawn.
[3]Applies to amounts in excess of $150,000 withdrawn in any year. This tax also has an impact on the determination of estate taxes at the taxpayer's death.

Minimum Distribution Tax

As the name implies, TDRAs were originally intended to provide income to retirees—not estates for their heirs. Thus, additional rules require that minimum withdrawals be made from TDRAs on an annual basis beginning no later than the year in which the taxpayer reaches age 70.5. The actual withdrawal for this first "required" year (the year in which age 70.5 is reached) may be deferred as late as April 1 of the following year if desired. This deferral provision can be advantageous since it also defers the taxes on the first "required" year's withdrawal for a year. However, since the taxpayer must also make a withdrawal for the second "required" year (the year in which age 71.5 is reached), effectively making *two* withdrawals in this second year could place some income into a higher tax bracket which might more than offset the benefit of deferring the taxes on the first "required" year's withdrawal (e.g., Katz, 1990, p. 56). At any rate, annual withdrawals must also be made in each of the following years (where ages 72.5, 73.5, . . . are reached).

The *actual* minimum amount that must be withdrawn in each year beginning at age 70.5 is determined as follows. For each separate TDRA the taxpayer owns, a *theoretical* minimum figure is calculated by dividing the balance in the account at the beginning of the year by the joint life expectancy of the taxpayer and the designated beneficiary for the account (e.g., Johnson, 1990). A schedule of life expectancy factors is supplied in IRS publication 590 (1992) to assist in determining this minimum amount. The minimum figure for each account is *theoretical* in that the IRS does not actually require the money to be withdrawn from this particular account, provided the sum of the *actual* withdrawals is at least as much as the sum of the *theoretical* figures (e.g., Geller, 1988; Solbee, 1988). For instance, Table 2 shows the calculations for a taxpayer with two TDRAs where the minimum required withdrawal is $72,172. This taxpayer may satisfy the IRS requirements by withdrawing *at least* $72,172 from either account or in any combination between accounts. Of course, the manner in which the taxpayer elects to split the withdrawal between these accounts will determine the amounts left in the accounts and, therefore, also affects the minimum required withdrawals in subsequent years.

For retirees who are age 70.5 or older and fail to make the minimum required withdrawal, a Minimum Distribution Tax penalty of 50 percent applies to the difference between the actual amount withdrawn and the minimum required withdrawal. For instance,

TABLE 2.
Minimum Distribution Tax Example

	Account 1		*Account 2*	
Age	*Balance*	*Joint Life Expectancy*[1]	*Balance*	*Joint Life Expectancy*[2]
70	$700,000	26.2	$900,000	19.8
71	?	25.3	?	19.0
72	?	24.4	?	18.2
73	?	23.5	?	17.3
74	?	22.7	?	16.5
⋮	⋮	⋮	⋮	⋮

Minimum Required Withdrawal (at age 70) $= \frac{\$700{,}000}{26.2} + \frac{\$900{,}000}{19.8} = \$72{,}172$

Notes: [1]Assuming 40 year old beneficiary
[2]Assuming 72 year old beneficiary

if the minimum required withdrawal is $100,000 and the taxpayer only withdraws $50,000, a nondeductible penalty tax of $25,000 (i.e., 0.50(100,000 – 50,000)) must be paid (e.g., McCommbe, 1989).

Excess Distributions Tax

In keeping with the philosophy that TDRAs should be used to provide retirement income, the Excess Distributions Tax is intended to discourage people from using TDRAs to amass or receive excessive retirement benefits. This tax imposes a penalty on the amount by which the total withdrawal from all TDRAs exceed $150,000 in any year. For any such "excess distribution" made prior to age 59.5 the tax is effectively five percent of the amount exceeding $150,000 (and the 10 percent Premature Distribution Tax applies to the entire amount withdrawn). For "excess distributions" made at or after age 59.5 the tax is 15 percent of the amount exceeding $150,000. Note that this tax applies even to those who are 70.5 or older and are required to make a "minimum withdrawal" (as discussed above) in excess of $150,000. For instance, if the minimum required withdrawal is $175,000 and the taxpayer withdraws this amount, he or she must pay an Excess Distribution Tax penalty of $3,750 (i.e., 0.15(175,000 – 150,000)) in addition to the normal income taxes which apply. Of course, if this taxpayer tries to avoid the Excess Distribution Tax by withdrawing only $150,000 the 50 percent Minimum Distribution Tax would levy a penalty of $12,500 (i.e., 0.50(175,000 – 150,000)).

Estate Taxes

An estate tax return must be filed if a taxpayer's gross estate exceeds $600,000 at the time of death. Generally, the balances in TDRAs are included in the valuation of one's estate at the time of death and are subject to normal estate taxes. However, the law allows special exclusions which effectively eliminate normal estate taxes on accounts where one's spouse is the beneficiary or where the remaining balances are left to qualified charitable organizations (e.g., IRS Publication 448, 1992).

The Excess Distributions Tax described above also plays a role in determining the taxes due on one's estate. Again, since TDRAs are only intended to provide retirement income for a given taxpayer, the tax law maintains that there should not be an "excessive accumulation" of funds left in these accounts at the taxpayer's death. Congress has decided that a "reasonable" amount to have in TDRAs at one's death should total no more than the present value of a $150,000 annuity for the remaining actuarial life expectancy of the taxpayer at the time of their death. Thus, one's estate must pay a 15 percent penalty tax on the amount by which the total value of the deceased's TDRAs exceed this "reasonable" amount. This "excess accumulation" tax applies to all TDRAs regardless of beneficiary designations.

III. A DIFFICULT DECISION

From the previous discussion it is clear that the question of how one should go about making withdrawals from TDRAs can be difficult. For many, it is probably challenging enough to make withdrawals that are within the IRS guidelines. However, if one attempts to make withdrawals that are not only "legal" but also maximize the value of one's benefits, the problem enters a new realm of difficulty. In either case, a number of practical questions must be addressed such as:

1) When should withdrawals begin?
2) How long should they continue?
3) How much should be withdrawn each year?
4) From which accounts should withdrawals be made?

Clearly, the Premature Distributions Tax can be avoided by not making withdrawals before age 59.5 and the Minimum Distribution Tax can be avoided by making the required minimum withdrawal beginning at age 70.5. Thus, with respect to the first question above we can generally say that withdrawals should begin no sooner than age 59.5 and no later than 70.5. A more specific answer would require consideration of an individual taxpayer's income needs. Similarly, with respect to the third question above we can generally say that the taxpayer should make *at least* the minimum required withdrawal beginning at age 70.5 in order to avoid the onerous Minimum Distribution Tax penalty. However, in some instances it may be necessary and/or wise to withdraw *more* than the minimum required amount.

The "best" or optimal answer to all of the questions listed above requires one to consider the *simultaneous* impacts of a number of subtle factors over a period of years. For instance, when deciding from which accounts to actually withdraw money one must consider the rates of return on the various accounts and the schedule of life expectancy factors which will apply to future balances in the accounts. One might intuitively sense that the optimal withdrawal policy would first involve making withdrawals from the accounts with the lowest rate of return. However, it is possible that the beneficiary designations on higher yielding accounts will, in subsequent years, impose higher minimum required withdrawals which might force more taxable income into higher tax brackets.

For instance, let us suppose that the rate of return on Account 1 in Table 2 is less than that the return on Account 2. If the required withdrawal for Year 1 is made from Account 1 this will obviously cause more money to accumulate in Account 2 which, in turn, will cause the minimum required withdrawal in subsequent years to be higher (due to the smaller joint life expectancy value on this account). Part of this higher minimum required withdrawal may fall into a higher tax bracket that may more than offset the higher earnings on this account.

Thus, in some situations it might actually be best to withdraw money from accounts earning the highest rates of return. Similarly, it is easy to believe that one should avoid making withdrawals for as long as possible (if not needed as current income) as this allows the investment to continue to earn interest and defer taxes. However, it is possible that in some cases withdrawals should begin before age 70.5 to help avoid exposure to the Excess Distribution Tax.

Simultaneously evaluating all these factors affecting the decision can quickly overwhelm us and prompt many to adopt heuristic withdrawal policies (e.g., Gould, 1988; McIntosh & Hollinrake, 1988; Quinn, 1988; Saftner & Fink, 1990; Sage, 1988; TIAA-CREF, 1991; Tritch, 1988). Two such policies are summarized below:

Minimal Withdrawal Policy: Withdraw the maximum of the minimum required by the IRS or the minimum needed to reach the desired level of retirement income. The actual withdrawal is made from the account(s) paying the lowest rate(s) of interest.

Proportional Withdrawal Policy: Same as above except the actual withdrawal is made from all the accounts in proportion to their balances at the beginning of the year. (Note: This is the default withdrawal policy used by TIAA–CREF.)

Notice that these policies ensure that the taxpayer withdraws at least as much as required by the IRS to avoid the 50 percent Minimum Distribution Tax penalty. While such policies may provide good "rules of thumb" for the average investor to follow, specific individuals can lose thousands of dollars by making suboptimal withdrawals using these heuristics (e.g., Katz, 1990; Ragsdale, Seila, & Little, 1993). If one considers the number of individuals with TDRA investments and the amount of money deposited in these accounts, the total potential loss to individual taxpayers as a result of poor or even slightly suboptimal decision making is considerable. Thus, even heuristics that generally work well still may leave specific individuals with *very* suboptimal withdrawal schedules.

IV. AN OPTIMIZATION MODEL

A taxpayer facing the decisions described above might be interested in determining the schedule of withdrawals that maximizes the net (after tax) present value (NPV) of the withdrawals made over their life expectancy plus the NPV of the remaining TDRA balances passing to their beneficiaries (all within IRS regulations). In this section, we present a mathematical programming model that can be solved to determine the schedule of withdrawals that achieves this objective.

Assumptions

Since our model considers a series of withdrawals over a number of years, it is clearly unrealistic to assume that the tax law will not change during this time. On the other hand, it is also clearly impossible to foresee what these changes will be—particularly those of a political or economic nature. However, other changes do seem reasonably certain. For instance, it is reasonable to expect the tax rate schedules to change every year to account for inflation. Today's tax rate schedules will almost certainly not apply 10 years from now. But we might expect the current schedules adjusted for inflation may reasonably estimate what may exist 10 years hence. Similarly, the $150,000 limit involved in today's Excess Distribution Tax and Estate Tax will almost certainly be adjusted for inflation in the future. Thus, our model accounts for inflationary changes in the tax law wherever appropriate. However, it does not account for unforeseen structural changes in the tax law. Such changes would have to be incorporated into the model as they occur.

A number of additional assumptions are also reflected in this model.

1. We assume that withdrawals are made at the end of each year to allow the taxpayer to accumulate as much tax-deferred income as possible. The formulation can easily be modified so that withdrawals are assumed to occur at the beginning of each year or on some other periodic basis, if so desired.
2. We assume that no contributions have been made to any accounts on a non-taxable basis. Additional variables and constraints, currently omitted for simplicity, can be added to our model to accommodate non-taxable contributions.

3. The effects of distributions on state income taxes and the taxability of social security benefits are ignored. No one model can incorporate all the various state tax laws and social security benefits would almost certainly be taxable (and immaterial) for those who would benefit significantly from this model.
4. We assume this model will be used for taxpayers who are at least 59.5 years old and that the Minimum Distribution Tax is to be avoided for those over the age of 70.5.

Discussion of The Model

A glossary of the terms used in our model is provided as a reference in an Appendix. With these terms and the previous assumptions in mind, the objective of our model may be stated as follows:

$$MAX \sum_{i=1}^{n} pv_i \left(\sum_{k=1}^{3} (1 - t_k^p) x_{ik} - 0.15 e_i \right) + pv_n \left(\sum_{j=1}^{n_l} b_{n+1,j} - \sum_{k=1}^{19} t_k^e y_k - 0.15 \varepsilon \right) \qquad (1)$$

This objective consists of two major present value elements. The first part of this objective calculates the present value of the taxpayer's total income each year after personal and "excess distribution" taxes, assuming the taxpayer is expected to live another n years. The second part of the objective calculates the present value of the expected balances in the taxpayer's TDRA accounts at the time of their death, less estate and excess accumulation taxes. Thus, the objective in equation (1) attempts to maximize the NPV of the withdrawals made over the taxpayer's life expectancy plus the NPV of the remaining TDRA balances passing to their beneficiaries. The meaning of the individual terms in this objective are described in an Appendix and will become more clear as the rest of the model is described.

A number of constraints are required to ensure that the terms in the objective function assume their proper values and that withdrawals are made within the IRS guidelines. However, an individual taxpayer might also be interested in ensuring that some minimum level of retirement income is available each year. Thus, constraint equations (2) and (3) are included in our model to ensure that the amounts withdrawn from the various TDRAs in year i (i.e., the w_{ij}) along with the taxpayer's other taxable income in year i (o_i) are sufficient to attain some minimum desired taxable retirement income in year i (d_i).

$$\sum_{j=1}^{n_l} w_{ij} + o_i \geq d_i, \quad i = 1, \ldots, n_r, n_{r+2}, \ldots, n \qquad (2)$$

$$\sum_{j=1}^{n_l} (w_{ij} + w'_{n_r j}) + o_i \geq d_i, \quad i = n_{r+1} \qquad (3)$$

The term $w'_{n_r j}$ in equation (3) represents the deferred withdrawals for the year in which the taxpayer turns age 70.5 (n_r). These withdrawals are deferred until April 1 of the following year (n_{r+1}). This deferral provision was described earlier in Section 2.2.

Constraint equations (4) and (5) ensure that beginning at age 70.5 (i.e., year n_r) the TDRA withdrawals made in each year are greater than or equal to the minimum required withdrawals imposed by IRS regulations. As illustrated in Table 2, these minimum required withdrawals are determined by dividing the balance in each account at the beginning of the year (b_{ij}) by the joint life expectancy factor (a_{ij}) supplied by the IRS.

$$\sum_{j=1}^{n_l} (w_{ij} + w'_{n_r j} - b_{ij}/a_{ij}) \geq 0, \quad i = n_r \tag{4}$$

$$\sum_{j=1}^{n_l} (w_{ij} - b_{ij}/a_{ij}) \geq 0, \quad i = n_{r+1}, \ldots, n \tag{5}$$

As mentioned in Section 2.3, amounts withdrawn from TDRAs in excess of the "reasonable withdrawal limit" (presently $150,000) in any year are subject to a 15 percent Excess Distributions Tax. Constraint equations (6) and (7) force the variable e_i to equal the amount by which withdrawals exceed the inflation-adjusted "reasonable withdrawal limit" ($\mathscr{L}_i$) in year i. Any such "excessive" withdrawals are then subjected to the 15 percent Excess Distribution Tax as shown in equation (1).

$$\sum_{j=1}^{n_l} w_{ij} - e_i \leq \mathscr{L}_i, \quad i = 1, \ldots, n_r, n_{r+2}, \ldots, n \tag{6}$$

$$\sum_{j=1}^{n_l} (w_{ij} + w'_{n_r j}) - e_i \leq \mathscr{L}_i, \quad i = n_{r+1} \tag{7}$$

In equations (8) and (9) the total taxable income from withdrawals (w_{ij}) and other sources (o_i) is allocated to the variables that represent the total income falling into each of the three different personal income tax brackets each year (i.e., x_{i1}, x_{i2}, and x_{i3}). Given the current personal tax rate structure, the objective function in equation (1) will ensure that taxable income will be first allocated to x_{i1} and then to x_{i2} and x_{i3} since the tax rates for each of these income brackets (represented by t_k^p) are monotonically increasing (i.e., $t_1^p < t_2^p < t_3^p$). Inflation-adjusted upper bounds (m_{ik}^p) for the income brackets are given in equation (10).

$$\sum_{k=1}^{3} x_{ik} - \sum_{j=1}^{n_l} w_{ij} - o_i = 0, \quad i = 1, \ldots, n_r, n_{r+2}, \ldots, n \tag{8}$$

$$\sum_{k=1}^{3} x_{ik} - \sum_{j=1}^{n_l} (w_{ij} + w'_{n_r j}) - o_i = 0, \quad i = n_{r+1} \tag{9}$$

$$x_{ik} \leq m_{ik}^p \quad i = 1, \ldots, n; k = 1, 2 \tag{10}$$

The constraints in equations (11) through (13) indicate that each account's balance at the beginning of each year (b_{ij}) should equal the prior year's beginning balance plus the interest earned, less any amount withdrawn from the account. Constraint equations (12) and

(13) apply, respectively, to the years in which the taxpayer reaches age 70.5 (n_r) and 71.5 (n_{r+1}). These constraints make the necessary adjustments to the beginning balances for years n_{r+1} and n_{r+2} if the deferral provision described in Section 2.2 is utilized. Notice that if the deferral provision is not used (i.e., if all $w'_{n_r j} = 0$) constraint equations (12) and (13) assume the same form as equation (11).

$$b_{i+1,j} - (1 + r_{ij})b_{ij} + w_{ij} = 0, i = 1, \ldots, n_{r-1}, n_{r+2}, \ldots, n; j = 1, \ldots, n_I \quad (11)$$

$$b_{i+1,j} - (1 + r_{ij})b_{ij} + w_{ij} + w'_{n_r j} = 0, i = n_r ; j = 1, \ldots, n_I \quad (12)$$

$$b_{i+1,j} - (1 + r_{ij})b_{ij} + w_{ij} - \tfrac{1}{4} r_{ij}w'_{n_r j} = 0, i = n_{r+1}; j = 1, \ldots, n_I \quad (13)$$

The remaining constraints in the model have to do with estate taxes. As mentioned earlier, an estate tax return does not have to be filed if the value of a taxpayer's estate does not exceed a certain cutoff point (presently \$600,000). We use $\mathcal{S}$ to represent this cutoff point (adjusted for inflation) and $\mathcal{E}$ to represent the estimated future value of the taxpayer's estate excluding TDRAs. Since more than one beneficiary can be designated for a given TDRA, it is also necessary to consider the percentage (p_j) of each TDRA which passes to the taxpayer's spouse or to a qualified charitable organization (which are both exempt from estate taxes) in determining the taxable value of an estate. In equation (14), the surplus variable s will equal the amount by which the taxpayer's gross taxable estate *exceeds* $\mathcal{S}$. Similarly, if estate taxes must be paid the optimal value for the binary variable λ will be one in equation (15) (where M represents a very large number).

$$\sum_{j=1}^{n_I} (1 - p_j)b_{n+1,j} + \mathcal{E} - s \leq \mathcal{S} \quad (14)$$

$$s \leq M\lambda \quad (15)$$

From equations (14) and (15) we know that if estate taxes must be paid, $\lambda = 1$ and the taxable value of the estate is $\mathcal{S}+s$. In this case, it is necessary to allocate the taxable value of the estate to variables in equation (1) which represented the different estate income tax brackets (i.e., the y_k). This is accomplished in equation (16). Upper bounds for the different estate income tax brackets are given in equation (17). Also notice that since the y_k are penalized in equation (1), an optimizer will attempt to set $s = 0$ and $\lambda = 0$ whenever possible. Thus, in equations (14) and (15), s and λ will assume strictly positive values *if and only if* estate taxes must be paid.

$$\sum_{k=1}^{19} y_k - s - \mathcal{S}\lambda = 0 \quad (16)$$

$$y_k \leq m^e_k, k = 1, \ldots, 18 \quad (17)$$

Given the current estate tax rate structure, the objective function in equation (1) will ensure that the taxable value of the estate will be first allocated to y_1 and then to y_2 and so forth up to y_{17} since the tax rates for each of these income brackets are monotonically increasing (i.e., $0.18 = t_1^e < t_2^e < \ldots < t_{17}^e = 0.55$). However, $t_{18}^e = 0.60$ and $t_{19}^e = 0.55$. Thus, when necessary, the objective in equation (1) would favor skipping y_{18} and allocating the remaining estate income to y_{19} since this has a lower tax rate. Constraint equations (18) and (19) ensure that this does not happen. It is clear from equation (19) that y_{19} can assume a positive value if and only if the binary variable $\bar{\lambda}$ is zero. However, if $\bar{\lambda} = 0$ then by equations (17) and (18) it must be that $y_{18} = m_{18}^e$. Thus, the inclusion of equations (18) and (19) in the model ensure that estate income is allocated to y_{19} if and only if *all* other y_k (including y_{18}) have reached their upper bounds.

$$y_{18} - m_{18}^e \bar{\lambda} \geq m_{18}^e \tag{18}$$

$$y_{19} - M(1 - \bar{\lambda}) \leq 0 \tag{19}$$

The last constraint in our model deals with the "excess accumulation" tax. The constraint in equation (20) forces the variable ε to equal the amount by which the ending TDRA account balances exceed the inflation-adjusted "reasonable" amount ($\mathscr{R}$) that a taxpayer should have in their TDRAs at the time of their death (see Appendix for a detailed definition of $\mathscr{R}$). Note that the "excess accumulations" represented by ε are subjected to a 15 percent tax in equation (1).

$$\sum_{j=1}^{n_l} b_{n+1,j} - \varepsilon \leq \mathscr{R} \tag{20}$$

Finally, bounds and integral conditions for the variables in this model are given in equations (21) and (22).

$$x_{ik}, y_k, e_i, \varepsilon, w_{ij}, w'_{n_r j}, b_{ij} \geq 0, \forall\ i, j, k \tag{21}$$

$$\lambda, \bar{\lambda} \in \{0, 1\} \tag{22}$$

In general, this model consists of $2n(n_l + 2) + 23$ variables and $n(6 + n_l) - n_r + 25$ constraints. In most real–world situations the formulation of this model would require several hundred variables and several hundred constraints. Since the model has only two integer (binary) variables and all the constraints are linear, a problem of this size would be fairly easy to solve using a personal computer and almost any commercial optimization package.

V. AN EXAMPLE

We shall illustrate the potential benefits of our model by analyzing the decision problem faced by the hypothetical taxpayer described below.

Example:

Suppose a 68 year old taxpayer has two TDRAs. The first account has a current value of \$2,000,000 and is invested in a money market fund yielding a 7.25% return. The taxpayer's 68 year old spouse is the beneficiary for this account. The second account has a current value of \$1,500,000 and is invested in a mutual fund yielding a 7.5% return. The

TABLE 3.
Data Items Required For Model Formulation

Personal Information	*Economic Information*	*Information Required For Each TDRA*
Present age of taxpayer and spouse	Discount rate	Current account balance
	Expected rate of inflation	Expected annual investment return
Life expectancy of taxpayer and spouse	Minimum desired level of annual retirement income (plus a minimum desired annual growth factor for the above amount)	Age of oldest beneficiary on each account
	Other Non-TDRA annual retirement income (plus an estimated annual growth factor for the above amount)	Percentage of account balance bequeathed to spousal beneficiary
	Expected value of tax payer's estate (excluding TDRAs)	Percentage of account balance bequeathed to charitable organizations

taxpayer's 35 year old daughter is the beneficiary for this account. The taxpayer has other taxable retirement income of $50,000 per year which is expected to increase with inflation at 4% each year and desires a minimum retirement income of $75,000 per year increasing by 5% per year. The taxable value of the taxpayer's estate is expected to be $2,000,000 (excluding TDRAs) at the time of his or her death. Finally, assume the taxpayer uses the "married filing jointly" tax rate schedule and that "risk free" investments are available yielding 5.25% annually.

While the model described above might appear somewhat daunting, it is important to realize that it can be made totally transparent to its user when implemented as part of a computerized decision support system. Such a system would provide a user–friendly interface to guide the user through entering the various data items shown in Table 3. The system would then automatically formulate and solve the model without any additional user intervention. Once the "optimal" schedule of withdrawals over the taxpayer's life expectancy has been determined the user would be provided with a summary analysis of the different withdrawal policies as described below.

Analyzing the Alternatives

There are two major issues for the taxpayer in our example to consider: the amount of money withdrawn from the accounts and the amounts that will be left to the beneficiaries. Thus, we first compare the results of the optimal withdrawal policy identified by our model to the two heuristic policies in terms of the NPV of the withdrawals and NPV of ending TDRA balances left at the taxpayer's death. These results for our example are given in Table 4.

As expected, the optimal withdrawal policy determined using the model results in the highest total NPV; in this case, over $900,000 more than the minimal withdrawal policy and over $340,000 more than the proportional policy. Interestingly, this solution provides not only the greatest amount of retirement income for the taxpayer (via withdrawals), but also results in the largest benefits (net of estate taxes) at the taxpayer's death. This, of course, is due to the relatively large amounts of estate taxes required to be paid under the heuristic withdrawal policies. To better understand the reason for this, consider the schedules of withdrawals and ending account balances presented in Tables 5 and 6.

TABLE 4.
Comparison of Present Values of Withdrawals & Ending Balances
(Original Example)

	Withdrawal Policy		
	Minimal	*Proportional*	*Optimal*
PV of Withdrawals	$2,544,496	$2,692,088	$2,740,709
PV of Income Taxes on Withdrawals	(842,932)	(910,824)	(933,190)
NPV of Withdrawals	$1,701,564	$1,781,264	$1,807,519
PV of Ending Balance	2,050,674	1,866,016	1,817,441
PV of Estate Taxes on Ending Balance	(1,204,765)	(532,270)	(168,560)
NPV of Ending Balance	$845,909	$1,333,746	$1,648,881
Total NPV	$2,547,473	$3,115,010	$3,456,400

In Table 5 we see that under the minimal withdrawal policy, withdrawals are first made from Account 1—the account with the lowest investment yield. Thus, as shown in Table 6 this policy causes the balance in Account 1 to decline steadily beginning at age 70 until it reaches zero. This leaves a substantial balance in Account 2. In contrast, Table 5 shows that the optimal policy follows the same withdrawal pattern as the minimal policy up through age 73 and afterward withdrawals begin to be made from Account 2. Thus, the optimal policy leaves a substantial balance in Account 1 and zero in Account 2 as shown in Table 6. Of course, the proportional policy withdraws money from both accounts over the years and leaves balances in each account at the taxpayer's death.

The reason for the relatively large amount of estate taxes for the heuristic withdrawal policies in Table 4 should now be clear. In this example, the taxpayer's spouse is the beneficiary on Account 1 and, as discussed above, a surviving spouse is effectively exempt from normal estate taxes on TDRAs. Thus, the estate taxes listed for the optimal withdrawal

TABLE 5.
Comparison of Withdrawal Amounts

	Minimal		*Proportional*		*Optimal*	
Age	*Account 1*	*Account 2*	*Account 1*	*Account 2*	*Account 1*	*Account 2*
68	$25,000	$0	$14,286	$10,714	$25,000	$0
69	26,750	0	15,270	11,480	26,750	0
70	175,237	0	100,070	75,405	175,237	0
71	186,513	0	106,871	80,726	186,513	0
72	199,036	0	114,497	86,696	199,036	0
73	211,598	0	122,239	92,785	211,598	0
74	224,358	0	130,271	99,123	156,705	67,652
75	238,171	0	139,032	106,048	0	239,169
76	252,615	0	148,352	113,435	0	257,483
77	266,123	0	157,304	120,576	0	275,690
78	281,678	0	167,777	128,921	0	297,121
79	295,873	0	177,702	136,886	0	318,314
80	310,246	0	188,102	145,257	0	341,165
81	324,652	0	198,979	154,039	0	365,870
82	338,192	0	209,152	162,319	0	390,030
83	350,305	0	220,401	171,478	0	418,322
84	363,087	0	231,126	180,276	0	446,094
85	9,963	362,674	241,423	188,785	0	475,444

TABLE 6.
Comparison of Ending Balances

	Minimal		*Proportional*		*Optimal*	
Age	*Account 1*	*Account 2*	*Account 1*	*Account 2*	*Account 1*	*Account 2*
68	$2,120,000	$1,612,500	$2,130,714	$1,601,786	$2,120,000	$1,612,500
69	2,246,950	1,733,438	2,269,921	1,710,440	2,246,950	1,733,438
70	2,234,617	1,863,445	2,334,420	1,763,319	2,234,616	1,863,445
71	2,210,113	2,003,204	2,396,795	1,814,842	2,210,112	2,003,204
72	2,171,311	2,153,444	2,456,066	1,864,259	2,171,309	2,153,444
73	2,117,133	2,314,953	2,511,891	1,991,293	2,117,130	2,314,953
74	2,046,267	2,488,574	2,563,732	1,955,517	2,113,917	2,420,922
75	1,956,450	2,675,218	2,610,571	1,996,133	2,267,176	2,363,323
76	1,845,677	2,875,859	2,651,485	2,032,408	2,431,546	2,283,089
77	1,713,366	3,091,549	2,686,413	2,064,262	2,607,833	2,178,631
78	1,555,907	3,323,415	2,713,401	2,090,161	2,796,901	2,044,907
79	1,372,838	3,572,671	2,732,420	2,110,037	2,999,676	1,879,961
80	1,162,128	3,840,622	2,742,419	2,123,033	3,217,153	1,679,793
81	921,725	4,128,668	2,742,265	2,128,222	3,450,396	1,439,908
82	650,358	4,438,319	2,731,927	2,125,520	3,700,550	1,157,870
83	347,205	4,771,193	2,709,591	2,113,455	3,968,840	826,388
84	9,290	5,129,032	2,674,911	2,091,689	4,256,581	442,274
85	0	5,151,036	2,627,418	2,059,781	4,565,183	0

policy in Table 4 represent only the 15 percent Excess Distribution Tax on balances exceeding the "reasonable" value one should have in TDRAs at the time of death. The estate taxes under the minimal and proportional policies include this Excess Distribution Tax *plus* the normal estate taxes due on the balances in Account 2 (with the taxpayer's child as beneficiary).

VI. DISCUSSION

The preceding example highlights a number of interesting issues. First, it is clear that the use of heuristics can lead to decisions that can result in very large financial losses. Of course, the example presented here is carefully chosen to illustrate this fact and one might argue that it is obvious that the minimal withdrawal policy will perform poorly in this situation. Indeed, other examples exist which show the minimal withdrawal policy to outperform the proportional policy and still other examples show that these heuristics can also be optimal. However, determining all the conditions under which these heuristics can be expected to perform well or poorly is, in itself, a difficult or impossible task considering the *infinite* combinations of beneficiary designations, life expectancies, and interest rate assumptions that one can encounter. Our model avoids this exercise by directly determining the optimal schedule of withdrawals under *any* set of conditions. Additionally, even in cases where the model only serves to confirm one's intuition about the optimal withdrawal schedule, there can be a significant psychological benefit in knowing with certainty that the best possible decision has been made for the given assumptions.

TABLE 7.
Comparison of Present Values of Withdrawals & Ending Balances
(Modified Example)

	Withdrawal Policy		
	Minimal	*Proportional*	*Optimal*
PV of Withdrawals	$2,892,660	$2,884,784	$2,892,660
PV of Income Taxes on Withdrawals	(1,003,087)	(999,464)	(1,003,087)
NPV of Withdrawals	$1,889,573	$1,885,320	$1,889,573
PV of Ending Balance	1,653,690	1,636,203	1,653,690
PV of Estate Taxes on Ending Balance	(143,998)	(141,375)	(143,998)
NPV of Ending Balance	$1,509,692	$1,494,828	$1,509,692
Total NPV	$3,399,265	$3,380,148	$3,399,265

Second, this example also highlights a number of withdrawal policy matters where one's intuition can fail. For instance, our example dramatically demonstrates how undesirable it can be to leave money in a TDRA with a non-spouse beneficiary. This might lead one to believe that, given the option, it *is* best to first make withdrawals from accounts with non-spouse beneficiaries. However, the schedule of withdrawals in Table 5 clearly indicates that this is not always the case since the optimal policy here involves first making withdrawals from Account 1 where the spouse is the beneficiary. Similarly, our example might lead one to believe that it is best to make one's spouse the beneficiary on all TDRAs so as to avoid the payment of normal estate taxes at one's death. (Note that this alternative may not always be possible since not all taxpayers are married and, even if they are, they may be unable to change the beneficiary designations on certain accounts.) The results for this scenario are presented in Table 7.

In Table 7 we see that if the spouse is the beneficiary on both accounts, the minimal and optimal withdrawal policies are identical while the proportional policy is only slightly suboptimal. Notice, however, that the NPV of the withdrawals under this scenario are larger than for the example in Table 4. This is due to the fact that the joint life expectancy of the taxpayer and his or her spouse is smaller than the joint life expectancy of the taxpayer and his or her child. This, in turn, forces the required minimum withdrawals that begin at age 70.5 to be larger. Thus, under this scenario the taxpayer receives $82,053 more in NPV during their lifetime via withdrawals, but the NPV being left to his or her spouse is reduced by $139,188. Thus, the overall effect of making the spouse the beneficiary on both accounts is to *decrease* the NPV of the taxpayer's total benefits by about $57,000 compared to the optimal solution in the original example.

Of course, there might be some who would prefer the solution offered by this second scenario even though the total NPV of the benefits are smaller. This illustrates yet another important aspect of the model. By allowing a person to easily play out such "What if?" scenarios, our model can determine not only the optimal withdrawal policy for a given set of conditions, but also allow the user to explore how changes in these conditions impact the solution. This can lead to the discovery of "better" (higher NPV) solutions. It can also lead to the discovery of solutions which have a smaller NPV but greater utility to individual decision makers. In either case, the model can provide the user with a greater understanding and sharper intuition about the problem they face and an objective means for assessing the trade-offs among the various possible decisions.

VII. CONCLUSIONS

In this paper we presented a mathematical programming model for assisting retirees (or their advisors) in determining how to make withdrawals from TDRAs. This model can be used to determine the withdrawal schedule that maximizes the NPV of the taxpayer's retirement benefits. The potential benefits of this model were illustrated relative to a number of heuristic withdrawal policies likely to be used in practice.

It is important to note that the use of this model is not intended to be a one-time occurrence. While the model determines the optimal schedule of withdrawals over a number of years, the taxpayer really is only immediately interested in what action they should take in the *current* year. Thus, the model can and should be updated on an annual basis to reflect changes in investment returns, life expectancies, beneficiary designations and, of course, structural changes in the tax law. When used in this manner our model offers the taxpayer the assurance of knowing what the best possible withdrawal decision is made each year based on the information at hand.

Finally, as mentioned at the outset, there are some exceptions to the tax rules embodied in our model that could have significant impacts on individual taxpayers. Thus, our model should not be used as a replacement for tax advisors but in tandem with qualified financial planning professionals.

APPENDIX

Glossary of Terms in the Model

a_{ij} life expectancy factor at year *i* for investment *j* (from IRS [1991b] tables).

b_{ij} balance at beginning of year *i* in TDRA investment *j*.

d_i desired minimum total taxable income from all sources in Year *i*.

δ the "risk free" discount rate.

$\mathscr{E}$ estimated total taxable future value of the taxpayer's estate (excluding TDRAs).

f the estimated rate of inflation.

$\mathscr{L}_i$ the "reasonable" withdrawal limit adjusted for inflation (i.e., $\mathscr{L}_i = \$150{,}000 \times (1+f)^i$).

e_i amount in excess of the "reasonable" withdrawal limit $\mathscr{L}_i$ withdrawn from TDRAs in Year *i*.

M an arbitrarily large positive number.

m_{ik}^p a constant representing the maximum amount of personal income allowed in personal tax bracket *k* in Year *i* (adjusted of inflation).

m_k^e a constant representing the maximum amount of estate income allowed in estate tax bracket *k* (adjusted for inflation).

n the life expectancy of the taxpayer at distribution Year 1 (or the year in which the first withdrawal is made).

n_r the distribution year in which the tax payer turns age 70.5.

n_I the number of TDRA investments.

o_i other (non-TDRA) taxable income in Year *i*.

p_j the percentage of the balance in TDRA *j* which at the taxpayer's death passes to his or her spouse or to a qualified charitable organization.

pv_i present value interest factor at year *i* (i.e., $pv_i = (1+\delta)^{-i}$).

r_{ij} the expected rate of return in Year i on TDRA investment j.
$\mathscr{R}$ the "reasonable" limit on the value of the taxpayer's TDRAs at the time of their death (i.e. $\mathscr{R} = \mathscr{L}_n \times (1 - (1 + \delta)^{-n_d})/\delta$ where n_d is the remaining actuarial life expectancy at the taxpayer's death.)
ε amount by which the final value of TDRAs exceed $\mathscr{R}$.
$\mathscr{S}$ the maximum amount of gross estate value allowed without having to file an estate tax return adjusted for inflation (i.e., $\mathscr{S} = 600{,}000 \times (1 + f)^n$).
t_k^p the k^{th} marginal personal income tax rate.
t_k^e the k^{th} marginal estate tax rate.
w_{ij} withdrawal in Year i from TDRA investment j.
$w'_{n_r j}$ withdrawal for year n_r from TDRA investment j made in year $n_r + 1$.
x_{ik} amount of total taxable income in Year i subject to tax rate t_k^p
y_k amount of total estate subject to tax rate t_k^e

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